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STABLE COMPLEX STRUCTURES ON REAL MANIFOLDS

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1. Introduction

1.1. An almost complex (a.c.) structure on a real differentiable manifold M^{2n} is a bundle automorphism J of its tangent bundle t such that $J^2 = -I$. Such a J provides a reduction of t to a complex vector bundle τ. It is natural to ask whether M carries the structure of a complex analytic manifold W^n whose tangent bundle is τ, or (a stronger requirement) whether a W exists for which J corresponds to multiplication by i in the fibers of τ. The stronger question is known as the integrability problem for J [3]. In general, both questions must be answered in the negative [1], [19]. However, the author has shown [2] that given J, the cartesian product $M^{2n} \times \boldsymbol{R}^{2k}$ always carries a complex structure W for some large integer k. In fact, we shall show that W is a Stein manifold. Thus, if we stabilize our problem, the weaker question has an affirmative solution. As we point out in the concluding section, our methods do not keep track of the a.c. structure J, and we cannot answer the stabilized version of the integrability problem. Our purposes in this note are to obtain bounds on the integer k and to classify, up to real deformations, the complex structures obtained from various reductions of t (or t stabilized) to complex vector bundles.

Our approach to the complexification problem is somewhat analagous to that used in solving the triangulation and smoothing problems [14], [9]. One first shows that reduction of the group of the tangent bundle of a manifold implies the existence of a suitable manifold structure on $M \times \boldsymbol{R}^k$ for large k. Then one invokes a splitting principle (in the smoothing problem, for example, this is the Cairns-Hirsch theorem) to show the new structure produced is equivalent to the cartesian product of M with appropriate new structure and $\boldsymbol{R}^k$ with its canonical structure. We have only a very limited splitting theorem for the complexification problem; namely, if M is known to support a Stein manifold structure W, our procedures yield essentially the cartesian products $W \times \boldsymbol{C}^k$. The results of [1] and [19] imply there is no general splitting theorem for complexifications.

1.2. This paper is divided in to six sections of which this is the first. In the second, we collect several facts from bundle theory which will be needed. The third section is devoted to a discussion of real deformation spaces of complex

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structures on noncompact manifolds and also contains the statement of our main result. The basic construction is described in section four and the result proved in section five. We conclude with a final section devoted to pertinent remarks and observations.

1.3. We now set down several conventions and some notation. All manifolds and diffeomorphisms between them will be assumed smooth of class C^∞. Our manifolds are Hausdorff, without boundary, and paracompact but not necessarily compact. If M^{2m} is a real $2m$-dimensional manifold which carries a particular structure of a complex analytic manifold, then M, considered as a complex manifold, will be denoted by W^m.

An $\boldsymbol{R}^k$ bundle $\boldsymbol{b}$ over M is a k-dimensional real vector bundle with total space $E(\boldsymbol{b})$ and structure group $O(k)$. Similarly, a $\boldsymbol{C}^k$ bundle β will be assumed to have structure group $U(k)$. In general, real bundles will be denoted by boldface letters and complex ones by Greek letters. We shall, when convenient, assume that all vector bundles over M are smooth and that any bundle equivalences which arise are also smooth. $\boldsymbol{e}^k$ will denote the trivial $\boldsymbol{R}^k$ bundle, and ε^k the trivial $\boldsymbol{C}^k$ bundle. When convenient, we shall identify $\boldsymbol{C}^k$ with $\boldsymbol{R}^{2k}$ and ε^k with $\boldsymbol{e}^{2k}$.

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2. Bundle theory

2.1. Let B be a topological space which has the homotopy type of a countable k-dimensional CW complex. If ξ is a $\boldsymbol{C}^n$ bundle over B, the natural inclusion of $U(n)$ in $O(2n)$ allows us to consider ξ as a $\boldsymbol{R}^{2n}$ bundle. Let $\boldsymbol{b}$ be a $\boldsymbol{R}^{2n}$ bundle over B.

Definition. i) *A complex reduction of* $\boldsymbol{b}$ is a pair (ξ, h), where ξ is a $\boldsymbol{C}^n$ bundle over B, and $h\colon \xi \to \boldsymbol{b}$ is an equivalence of $\boldsymbol{R}^{2n}$ bundles.

ii) Two reductions, (ξ_i, h_i), $i = 0, 1$, of $\boldsymbol{b}$ are *equivalent* if h_0 is isotopic to an equivalence $h\colon \xi_0 \to \boldsymbol{b}$ such that $h_1^{-1} \circ h\colon \xi_0 \to \xi_1$ is an equivalence of $\boldsymbol{C}^n$ bundles.

As in [15] we have

Proposition 1. *The equivalence classes of complex reductions of* $\boldsymbol{b}$ *are in one-one correspondence with the isotopy classes of lifts to* $BU(n)$ *of a classifying map* $f\colon B \to BO(2n)$ *for* $\boldsymbol{b}$.

Recall that two lifts of f are *isotopic* if they are homotopic via a homotopy $\{f_t\}$ such that f_t is a lift of f for each $t \in I$. The set of equivalence classes of complex reductions of $\boldsymbol{b}$ will be denoted by $CR(\boldsymbol{b})$. $[\xi, h]$ is the member of $CR(\boldsymbol{b})$ determined by a reduction (ξ, h).

It is a standard fact from homotopy theory that the inclusions $i\colon U(n) \to U(m)$ and $j\colon O(2n) \to O(2m)$ induce isomorphisms $i_*\colon \pi_k(BU(n)) \to \pi_k(BU(m))$ and $j_*\colon \pi_k(BO(2n)) \to \pi_k(BO(2m))$ when $2m > 2n > k$, where the map between classifying spaces determined by a map of groups is denoted by the same letter. Now, if $f: B \to BO(2n)$ classifies an R^{2n} bundle b, then $j \circ f$ classifies $b \oplus e^{2(m-n)}$. The isomorphisms in homotopy mentioned above, when combined with standard obstruction theory, can be applied to show that if $2n - 1 > k$ and F': $B \to BU(m)$ is any lift of $j \circ f$, then there exists a lift F: $B \to BU(n)$ of f such that $i \circ F$ is isotopic to F'. Further, if F'_i: $B \to BU(m)$ are isotopic lifts of $j \circ f$, $i = 1, 2$, then the lifts F_i: $B \to BU(n)$, $i = 1, 2$, which they determine, can be constructed so as to be isotopic lifts of f. Thus, if $2n - 1 > k$ and $l > 0$, then the function χ: $CR(b) \to CR(b \oplus e^{2l})$ defined by $\chi[\xi, h] = [\xi \oplus \varepsilon^l, h \oplus id]$ is a bijection. We shall, when convenient, identify there sets by means of χ. All this can be reformulated as follows: if $2n - 1 > k$ and $[\xi, h] \in CR(b)$, then ξ and b are *stable* bundles over B and are completely determined by elements $\{\xi\} \in \widetilde{KU}(B)$ and $\{b\} \in \widetilde{KO}(B)$. If π: $\widetilde{KU}(B) \to \widetilde{KO}(B)$ is the natural homomorphism induced by the inclusion of U in O, then $[\xi, h] \in CR(b)$ if and only if $\pi\{\xi\} = \{b\}$. The element $[\xi, h]$ is completely determined by $\{\xi\}$, and $CR(b)$ corresponds to $\pi^{-1}\{b\}$, a coset of $\ker \pi$. Thus, if we choose some element of $CR(b)$ as the neutral element, the correspondence with $\ker \pi$ allows us to define an abelian group structure on $CR(b)$ in a manner similar to that in [9].

2.2. If $2m - 1 > k$, and b, $\{b\}$ are as above, then we can find an R^{2m} bundle c over B with $\{c\} = -\{b\} \in \widetilde{KO}(B)$. Then, for each $\{\xi\} \in \pi^{-1}\{b\}$, $-\{\xi\} \in \pi^{-1}\{c\}$ and conversely, so there is a well-defined bijection between $CR(b)$ and $CR(c)$. We shall be particularly interested in the case when $b =$ the tangent bundle t of a real $2n$-dimensional manifold M^{2n} and $c =$ the stable normal R^{2m} bundle n for M, $2m \geq 2n + 2$. An almost complex structure on M determines a reduction (η, h) of t. This gives rise to a reduction $(\eta \oplus \varepsilon', h \otimes \mathrm{id})$ representing $\chi[\eta, h]$. But now the bundles are stable, so we have a well-defined element of $CR(n)$. We shall use this correspondence in section five.

3. Deformation and Stein Structures

3.1. Let S and V be real manifolds of dimensions s and $2n + s$ respectively, and ρ: $V \to S$ be a smooth surjection. The triple (V, S, ρ) will be said to have the structure of a *smooth family of complex structures* if there exists a family of quadruples $\{(N_j, U_j, W_j, h_j)\}_{j \in J}$ such that for each $j \in J, N_j$ is an open subset of V, U_j is an open subset of S, W_j is an open subset of C^n and h_j: $N_j \to U_j \times W_j$ is a diffeomorphism, all satisfying the conditions: i) $\{N_j\}_{j \in J}$ is an open cover of V which is an atlas for the differential structure of V; ii) $\rho | N_j = \pi_1 \circ h_j$ for each $j \in J$ where π_1 is projection to the first factor; iii) if $i, j \in J$ and $N_i \cap N_j$ is nonempty, then for each $x \in U_i \cap U_j$, the map $h_j \circ h_i^{-1}$: $h_i(N_i \cap N_j \cap \rho^{-1}(x)) \to h_j(N_i \cap N_j \cap \rho^{-1}(x))$ is a holomorphic map from an open subset of $\{x\} \times W_i$ to an open subset of $\{x\} \times W_j$.

It is an immediate consequence of iii) that for each $x \in S$, $M_x = \rho^{-1}(x)$ is a complex analytic n-dimensional manifold. If S is connected and for each $s \in S, M_x$ is known to be compact, it can be shown that all the M_x's, when considered as real $2n$-dimensional manifolds, are diffeomorphic; in fact, V is the space of a smooth bundle over S with fiber M and projection ρ. This may not be the case if some M_x is noncompact. However, this property holds for all families of deformations of interest in this paper. Consequently we shall require that our families (V, S, ρ) satisfy iv) there exists a real $2n$-dimensional namifold M such that (V, S, ρ) is a smooth bundle with fiber M. We shall be concerned with families in which $S = \boldsymbol{R}^1$. Consequently V is diffeomorphic to $\boldsymbol{R}^1 \times M$, and ρ is projection to the first factor.

3.2. A smooth family of complex structures of the form $(V, \boldsymbol{R}^1, \rho)$ will be called a *path of complex structures on M* where M is the real $2n$-dimensional manifold provided by condition iv) of the definition. If $C(M)$ is the set of all complex-manifold structures on M^{2n}, two elements $W_0, W_1 \in C(M)$ are said to be in the same path component if there is a path $(V, \boldsymbol{R}^1, \rho)$ of complex structures on M, with $M_t = \rho^{-1}(t)$ holomorphically homeomorphic to $W_t, t = 0, 1$. This is an equivalence relation. The reflexive property is obvious, and symmetry is seen to hold when one replaces the path $(V, \boldsymbol{R}^1, \rho)$ joining W_0 to W_1 by the path $(V, \boldsymbol{R}^1, \rho')$ where $\rho'(v) = 1 - \rho(v)$. To check transitivity, let $W_0, W_1, W_2 \in C(M)$, and let $(V_1, \boldsymbol{R}^1, \rho_1)$, $(V_2, \boldsymbol{R}^1, \rho_2)$ be paths of complex structures on M with holomorphic homeomorphisms $r_t : \rho_1^{-1}(t) \to W_t$, $t = 0, 1$ and $s_t : \rho_1^{-1}(t) \to W_{t+1}$, $t = 0, 1$. Pick a diffeomorphism D from $\boldsymbol{R}^1$ to itself which leaves fixed the points 0 and 1 and all of whose derivatives vanish at these points. Now, set $\sigma_i = D \circ \rho_i$, $i = 1, 2$, and let $V = \sigma_1^{-1}(-\infty, 1] \cup \sigma_2^{-1}[0, \infty)$ where $\sigma_1^{-1}(1) = \rho_1^{-1}(1)$ is identified to $\rho_2^{-1}(0) = \sigma_2^{-1}(0)$ by $s_0^{-1} \circ r_1$. Define $\rho_3 : V \to \boldsymbol{R}^1$ by $\rho_3 | \sigma_1^{-1}(-\infty, 1] = \frac{1}{2}\sigma_1$ and $\rho_3 | \sigma_2^{-1}[0, \infty) = \frac{1}{2}(1 + \sigma_2)$. Then $(V, \boldsymbol{R}^1, \rho)$ is a path of complex structures from W_0 to W_2.

3.3. For $n > 0$, a complex manifold W^n is a Stein manifold if and only if it can be holomorphically embedded as a closed subset of $\boldsymbol{C}^k$ for some large integer k, [6]. Necessarily, W is noncompact. If M is a noncompact real $2n$-dimensional manifold, we shall denote the subset of $C(M)$ consisting of the Stein manifold structures on M by $S(M)$. If $S(M)$ is nonempty, then M must be of the homotopy type of a CW complex of dimension $\leq n$ [17, p. 39]. A path $(V, \boldsymbol{R}^1, \rho)$ of complex structures on M is a *Stein path* if for each $t \in \boldsymbol{R}^1$, $M_t = \rho^{-1}(t)$ is a Stein manifold. As in § 3.2, $S(M)$ decomposes into path components, where two complex structures $W_0, W_1 \in S(M)$ are in the same path component of $S(M)$ if there is a Stein path joining them. $PS(M)$ denotes the set of path components of $S(M)$. We define $PS_k(M)$ to be $PS(M \times \boldsymbol{R}^{2k})$, $k \geq 0$. If W is a Stein manifold, so is the cartesian product $W \times \boldsymbol{C}^k$ for any $k \geq 0$. Clearly, if $(V, \boldsymbol{R}^1, \rho)$ is a Stein path joining W_0 to W_1 then $(V \times \boldsymbol{C}^k, \boldsymbol{R}^1, \rho \circ \pi_1)$ is a Stein path joining $W_0 \times \boldsymbol{C}^k$ to $W_1 \times \boldsymbol{C}^k$, and we have a natural function $\sigma : PS(M) \to PS_k(M)$. Let $\Pi\Sigma(M) = \lim_{k \to \infty} PS_k(M)$. Observe that if M^{2n} is compact, $PS(M)$ will be empty but for $k \geq 2n$, $PS_k(M)$ may very well be nonempty. So $\Pi\Sigma(M)$ is well defined for all even dimensional manifolds M. If M is odd-dimensional, we can define $PS_k(M)$ to be $PS(M \times \boldsymbol{R}^{2k+1})$, and again have $\Pi\Sigma(M)$ well defined.

3.4. We can now state our results. Let M be a real $2n$-dimensional manifold of the homotopy type of a k-dimensional CW complex, and $2m$ be the smallest even integer larger than k.

Theorem 1. *If $\boldsymbol{n}$ is the stable normal R^{2l} bundle of M, $2l \geq 2m > k$, there is a one-one correspondence between $CR(\boldsymbol{n})$ and $\Pi\Sigma(M)$.*

Corollary. *Let J be an almost complex (a.c.) structure on M, and J_1 be the canonical a.c. structure on $C^l = \boldsymbol{R}^{2l}$. Let $l \geq m + n$. Then $J \oplus J_1$, an a.c. strucrure on $M \times \boldsymbol{R}^2$, is homotopic to an integrable a.c. structure which integrates to a Stein manifold.*

The proofs of these statements appear in section five.

4. The main construction

In this section, we review and improve on the dimensional restrictions of a construction described by the author in [2]. Landweber has also considered these improvements on dimension in a preliminary version of [12].

4.1. Let M be a real n-dimensional manifold; we do not necessarily require n to be even. A classical result of Whitney allows us to assume M is a real-analytic manifold. Now for each real-analytic n-manifold M Whitney and Bruhat [20] have constructed a pair $(\tilde{M}, i)$, called a *complexification* of M, consisting of a complex n-dimensional manifold $\tilde{M}$ and a real analytic embedding $i: M \to \tilde{M}$ such that $i(M)$ is covered by a family $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ of coordinate charts in $\tilde{M}$ with the property that $h_\alpha(U_\alpha \cap i(M)) = \boldsymbol{R}^n \cap h_\alpha(U_\alpha) \subset \boldsymbol{C}^n$ for each $\alpha \in A$. Moreover, the germ of the complexification is unique in the sense that if $(\tilde{M}_j, i_j)$, $j = 1, 2$, are two complexifications of M, there exist neighborhoods U_j of $i_j(M)$ in $\tilde{M}_j$, $j = 1, 2$, and a holomorphic equivalence $f: U_1 \to U_2$ such that $f \circ i_1 = i_2$. The normal $\boldsymbol{R}^n$ bundle $\boldsymbol{n}$ to the embedding i of M in $\tilde{M}$ is thus independent of $(\tilde{M}, i)$, and is naturally equivalent to the tangent bundle $\boldsymbol{t}$ of M. Furthermore, Grauert [5] has shown that for any complexfication $(\tilde{M}, i)$ of M there is an open tubular neighborhood N of $i(M)$ in $\tilde{M}$ which is a Stein manifold. We shall identify N with the total space $E(\boldsymbol{t})$ of the tangent bundle of M via a smooth bundle equivalence, and let $r: N \to i(M)$ be the neighborhood retraction which corresponds under this identification to the projection of $\boldsymbol{t}$.

Suppose now that $n = 2m$, and M carries a complex m-manifold structure W with $\boldsymbol{C}^m$ tangent bundle τ. Let $\overline{W}$ be the conjugate complex structure on M with tangent bundle $\bar{\tau}$, the conjugate bundle to τ (considered as $\boldsymbol{C}^m$ bundles over M). Then we can choose as a complexification of M the pair $(W \times \overline{W}, d)$ where $d: M \to W \times \overline{W}$ $(= M \times M)$ is the diagonal map. The normal bundle to the embedding d is now a C^m bundle equivalent to $\bar{\tau}$; thus we can identify a tubular neighborhood N of $d(M)$ with $E(\bar{\tau})$ so that the retraction r onto $d(M)$ is projection to the first factor W. Observe that this identification is one of smooth manifolds; it need not preserve complex structures. In fact, if $(\tilde{M}, i)$ is any complexification of a real manifold M, we use the smooth identification of $E(\boldsymbol{t})$ with N to *define* a complex structure on $E(\boldsymbol{t})$. It is the structure pulled back by the equivalence from the Stein structure on N determined by Grauert. If M carries W as above, then the natural diffeomorphism of $E(\bar{\tau})$ and $E(\boldsymbol{t})$ gives a Stein structure on $E(\bar{\tau})$ since $\bar{\tau}$ reduces $\boldsymbol{t}$. This is the structure preserved by the equivalence, but is not, in general, the natural complex structure on $E(\bar{\tau})$ determined by $\overline{W}$.

4.2. Once more, M is a real n-dimensional manifold. Let $\boldsymbol{n}$ be a stable normal $\boldsymbol{R}^{2m}$ bundle for M, $2m > n$. Thus $\boldsymbol{t} \oplus \boldsymbol{n} = \boldsymbol{e}^{n+2m}$. For any $\xi \in CR(\boldsymbol{n})$, we can conclude that $\bar{\xi} \in CR(\boldsymbol{n})$ and $r * \bar{\xi} \in CR(r * \boldsymbol{n})$. Since N is a Stein manifold and $r * \bar{\xi}$ is a C^m bundle over N, it follows from another result of Grauert [4] that $r * \bar{\xi}$ may be considered in a canonical way as a holomorphic bundle over N. This determines a Stein manifold structure on $E(r * \bar{\xi})$ [6, p. 258]. Since N and $E(\boldsymbol{t})$ are equivalent as real manifolds, the underlying real structure on $E(r * \bar{\xi})$ is diffeomorphic to $E(r * \boldsymbol{n}) = E(\boldsymbol{t} \oplus \boldsymbol{n}) = E(\boldsymbol{e}^{n+2m}) = M \times \boldsymbol{R}^{n+2m}$. Thus to each reduction ξ of $\boldsymbol{n}$ we can assign a Stein structure W on $M \times \boldsymbol{R}^{n+2m}$. Actually, a reduction of $\boldsymbol{n}$ was defined to be a pair (ξ, h). A different choice (ξ_1, h_1) within the same equivalence class in $CR(\boldsymbol{n})$ would give $\xi = \xi_1$ and h isotopic to h_1. The isotopy gives rise to a Stein path joining W to the structure W_1 determined by (ξ_1, h_1). Thus we have a well-defined function $\gamma\colon CR(\boldsymbol{n}^{2m}) \to PS_{n+2m}(M)$. We pause to assure the reader that the reason for employing $\bar{\xi}$ in place of ξ will become evident in the next section.

4.3. Recall that if M has the homotopy type of a k-dimensional CW complex, then $\chi\colon CR(\boldsymbol{n}) \to CR(\boldsymbol{n} \oplus \boldsymbol{e}^{2l})$ is an equivalence when $2m > k$ and $l > 0$. On the other hand, it is easy to see that if the reduction (ξ, h) of $\boldsymbol{n}$ gives rise to the Stein structure W on $M \times \boldsymbol{R}^{n+2m}$, then the construction of § 4.2, when applied to $(\xi \oplus \varepsilon^l, h \oplus \text{id})$, gives rise to $W \times C^l$. So we have a commutative square

$$\begin{array}{ccc} CR(\boldsymbol{n}) & \xrightarrow{\gamma} & PS_{n+2m}(M) \\ \approx\big\downarrow\chi & & \big\downarrow\sigma \\ CR(\boldsymbol{n} \oplus \boldsymbol{e}^{2l}) & \xrightarrow{\gamma} & PS_{n+2(m+l)}(M) \end{array}$$

and a well-defined function $\Gamma\colon CR(\boldsymbol{n}) \to \Pi\Sigma(M)$.

5. Proofs

For the remainder of the argument, we shall assume M is a real even-dimensional manifold. Let $\dim M = 2m$.

5.1. Our first order of business is to construct an inverse to Γ. Let X be a Stein structure on $M \times R^{2l}$ for some $l \geq 0$. Then there is a holomorphic embedding $j: X \to C^{2(m+l)+1}$ with a holomorphic normal C^{m+l+1} bundle ξ [6, p. 256]. Observe that ξ is a stable bundle. Then ξ is a reduction of the stable normal bundle of M. More precisely, we note that some tubular neighborhood U of $j(X)$ can be holomorphically identified with the total space $E(\xi')$ of a disk sub-bundle of ξ [6, p. 257]. Using a radial projection in each fiber, we can identify $E(\xi)$ with $E(\xi')$ and thus with U, this identification being C^∞. It is a standard fact of differential topology that U can be identified with $E(\boldsymbol{n})$, [16]. Composing these identifications, and restricting our attention to those parts of the respective bundles over $M \times \{0\}$, we have our reduction. Clearly, if we follow j by some embedding of $C^{2(m+l)+1}$ into $C^{2(m+l)+1+p}$ the resulting reduction of $\boldsymbol{n} \oplus \boldsymbol{e}^{2p}$ is $\xi \oplus \varepsilon^p$. If $j': X \to C^k$ is some other holomorphic embedding of X, we can consider j and j' to be embeddings of X in general position in some C^p by embedding $C^{2(m+l)+1}$ and C^k in C^p as affine subspaces in general position. Now the embedding $J: X \times I \to C^p$ given by $J(x, t) = (1-t)j(x) + tj'(x)$ is holomorphic for each t-level and is an isotopy between j and j'. It provides an isotopy between the stable reductions of $\boldsymbol{n}$ determined by j and j'. Thus X determines a well-defined element in $CR(\boldsymbol{n})$. We now observe that the embedding theorem of [6, p. 224] can be modified to produce a level-preserving smooth embedding $J: V \to C^{2(n+l)+1} \times R^1$ of any Stein path (V, R^1, ρ) such that $J: \rho^{-1}(t) \to C^{2(n+l)+1} \times \{t\}$ is holomorphic for each real t. Thus reductions of $\boldsymbol{n}$ determined by different X's within the same Stein path class are equivalent. Hence we have a well-defined map $\Delta: \Pi\Sigma(M) \to CR(\boldsymbol{n})$.

5.2. We now show that $\Gamma\Delta$ is the identity map of $\Pi\Sigma(M)$. Let τ be the C^{n+l} tangent bundle of a Stein structure X on $M \times R^{2l}$, and ξ be the stable C^p normal bundle to X, p large. Then $\Delta(X)$ is represented by $\xi_1 = \xi \mid M \times \{0\}$. In carrying out the construction of § 4.2 to obtain $\Gamma(\xi)$ we note that we can take a tubular neighborhood of the diagonal in $X \times \bar{X}$ as N, and we can identify N with $E(\bar{\tau})$ (cf. § 4.1). Moreover, we can take $E(r * \bar{\xi}_1) \to N \to X$ to be a holomorphic bundle. But this bundle is equivalent, in a C^∞ sense, to $\bar{\tau} \oplus \bar{\xi}_1 = \overline{\tau \oplus \xi_1} = \bar{\varepsilon} = \varepsilon$. By [4], our bundle is also holomorphically trivial, so $\Gamma\Delta(X) = \{X \times C^{n+l+p}\} = \{X\}$ in $\Pi\Sigma(M)$.

5.3. We complete the proof of the theorem by showing that $\Delta\Gamma$ is the identity map of $CR(\boldsymbol{n})$. Let $\xi \in CR(\boldsymbol{n})$. We can assume that the real tangent bundle $\boldsymbol{t}$ of M is stable for M can be replaced by $M \times R^2$ in this argument. Now ξ determines a unique $\tau \in CR(\boldsymbol{t})$ with $\xi \oplus \tau = \varepsilon$.

It is a standard fact that if $(\tilde{M}, i)$ is a complexification of M with complex tangent bundle λ, then $\lambda \mid M = \boldsymbol{t} \otimes_R C$ which we can identify with $\tau \oplus \bar{\tau}$. Let $X = E(r * \bar{\xi})$ be the Stein structure obtained from ξ by the construction of § 4.2. X is a bundle over a complexification N of M. If μ is the tangent bundle of X then, regarding N as the zero-section of $r * \bar{\xi}$, $\mu \mid N = r * \bar{\xi} \oplus \lambda$. Thus $\mu \mid M = \bar{\xi} \oplus \bar{\tau} \oplus \tau = \tau \oplus \varepsilon$. If we now embed X in C^p, and let ζ be the normal bundle to the embedding, then $\mu \oplus \zeta = \varepsilon$; so $\tau \oplus \varepsilon \oplus \zeta | M = \varepsilon$. Thus $\zeta | M$ is a stable inverse to τ. But ξ is also a stable inverse to $\tau \in \widetilde{KU}(M)$. Since $\widetilde{KU}(M)$ is a group, $\xi = \zeta | M$. But $\zeta | M$ represents $\Delta\Gamma(\xi)$, so $\Delta\Gamma$ is the identity map.

5.4. To prove the corollary, we first observe that $J \oplus J_1$, when restricted to that part of the tangent bundle of $M \times \boldsymbol{R}^{2l}$ which lies over M, provides a complex reduction of the stable tangent bundle of M. This, in turn, induces a reduction of the stable normal bundle of M. The dimensional restrictions are such that the main construction gives a Stein structure on $M \times \boldsymbol{R}^{2l}$ with tangent bundle equivalent to the complex bundle determined by $J \oplus J_1$.

6. Remarks

6.1. There is a relative version of our existence result whose proof is a consequence of the material in §§ 4 and 5.

Theorem 2. *Let M^{2n} be a smooth unbounded real manifold, and V^{2k} a noncompact submanifold which carries a Stein manifold structure X^k. Assume the normal bundle $\boldsymbol{n}$ of V in M has been reduced to a $\boldsymbol{C}^{n-k}$ bundle ν. If ξ is a $\boldsymbol{C}^{n+1}$ bundle over M which reduces the stable tangent bundle $\boldsymbol{t} \oplus \boldsymbol{e}^2$ of M such that $\xi | V$ splits as $\tau(X) \oplus \nu \oplus \varepsilon$, then $M^{2n} \times \boldsymbol{R}^{4n+4}$ carries a Stein structure W which contains $X \times \boldsymbol{C}^{2n+2}$ as a holomorphic submanifold.*

6.2. Our approach to the whole problem of existence and classification of complex structures is essentially topological. It is crucial to our arguments that given almost complex structures can be changed within a homotopy class. Thus we have not tackled the classical integrability problem at all. To illustrate the variance of our approach from the classical problem, we point out that even if we begin with an a.c. structure which is known to be integrable, the stable complex structure obtained from our construction bears no immediate relation to the complex structure with which we began.

Proposition 2. *Let W^n be a complex structure on M^{2n}, with stable normal $\boldsymbol{C}^{n+1}$ bundle ν. Then $\gamma(\nu)$ is in the Stein path component of $W \times \boldsymbol{C}^{2n+1}$ if and only if W is a Stein manifold.*

One half of the proof is the content of § 5.2 where it is shown that $\Gamma\Delta$ is the identity of $\Pi\Sigma(M)$. For the other direction, it suffices to observe that W is a submanifold of $W \times \boldsymbol{C}^{2n+1}$. However a submanifold of a Stein manifold X must itself be Stein since any holomorphic embedding of X in some $\boldsymbol{C}^p$ induces an embedding of the submanifold.

On the other hand, it follows from work of Haefliger and Landweber, discussed below, that $\gamma(\nu)$ *is* in the same path component as $W \times \boldsymbol{C}^{2n+1}$ for any complex manifold W.

Finally, we note that the introduction of the limiting process in the definition of $\Pi\Sigma(M)$ is necessitated by a lack of more specific knowledge of the map

$\sigma: PS_k(M) \to PS_{k+1}(M)$. One conjectures σ is a bijection. This would be equivalent to a weak form of a splitting theorem such as the Cairns-Hirsch theorem. It would suffice to show σ is either one-one or onto. With respect to the "kernel" of σ, we have the following analogue of a theorem of Hirsch and Mazur in the topological, piecewise-linear and smooth categories [8].

Proposition 3. *Let W_1 and W_2 be Stein structures with the same underlying real manifold M^{2n} and equivalent stable tangent bundles. Then, for any $l \geq 2n + 1$, $W_1 \times C^l$ and $W_2 \times C^l$ are in the same Stein path component.*

6.3. We conclude with a remark on the relation of this paper to the work of Haefliger [7] and Landweber [12]. Haefliger has constructed a space $B\Gamma_n^C$ and a map $v: B\Gamma_n^C \to BU(n)$ such that for any real open $2n$-dimensional manifold M whose tangent bundle can be reduced to a C^n bundle τ classified by $\tau_c: M \to BU(n)$, M supports a complex structure with tangent bundle τ if and only if τ_c can be factored as $v\tau'$. Furthermore, homotopy classes of maps τ' (homotopic as lifts of τ_c) correspond to "integrable homotopy classes of complex structures". A close examination of the definitions shows that Haefliger's integrable homotopy classes are exactly our paths of complex structures; Stein structures do not enter here. Landweber has shown that the fiber of v is $(n-1)$-connected, using methods which have their genesis in Haefliger's approach and in our main construction [2]. An appeal to obstruction theory yields

Theorem (*Landweber*). *Let M be an open manifold of dimension $2n$. If $H^i(M: Z) = 0$ for $i > n$, then there is a natural bijection between paths of complex structures on M and homotopy classes of almost complex structures on M.*

Since we can replace M by $M \times R^{2l}$ in Landweber's theorem and obtain the same classification, we see that a weak form of the splitting theorem holds: *if M is as in Landweber's theorem, any complex structure on $M \times R^{2k}$ can be joined by a path to a structure of the form $W \times C^k$ where W is a complex structure on M.*

We note that if M is as in Landweber's theorem, its tangent R^{2n} bundle is stable. Now, Landweber's classification is in terms of $CR(t)$ which is naturally equivalent to $CR(n)$. Thus we may conclude that for any real $2n$-dimensional manifold M, any complex structure on $M \times R^{2k}$, $k > n$, may be deformed within its path component to a Stein structure. This leads is to conjecture that if M is as in Landweber's theorem, any complex structure on M can be deformed within its path component, to a Stein manifold. We conclude with the observation that Landweber's hypotheses on M are necessary if M carries a Stein structure [17, p. 39].

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